Higher Mathematics – Course Notes

Equation of a Line

(i) Collinearity: If points are collinear then they lie on the same straight line.
i.e. to show that A, B and C are collinear, show that
$\mathbf{m_{AB} = m_{BC}}$ **and state that B is a common point**.

(ii) Gradient: Can be found in 4 ways depending on the information given.

(1) From the equation of a line. The equation **must** be in the form $\mathbf{y = mx + c}$
e.g. $3y - 6x + 9 = 0 \rightarrow y = 2x - 3$ i.e. gradient is 2.

(2) Given two points. Use $\mathbf{m} = \dfrac{y_2 - y_1}{x_2 - x_1}$

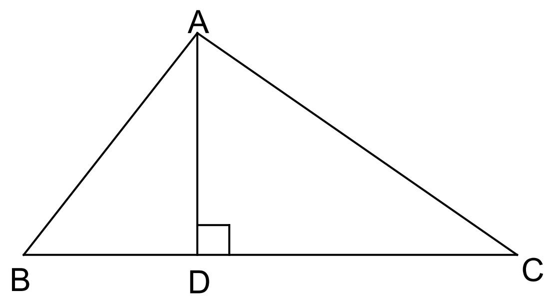
(3) Given an angle. Use $\mathbf{m = \tan\theta}$
Note: Angle used **must** be with positive direction of x-axis.

(4) Given a function(or equation). Use differentiation.

(iii) Types of line: To find the equation of a line you need **a point on the line and the gradient of the line.**

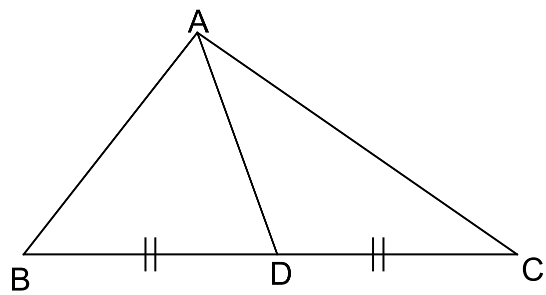
Altitude From the corner of a triangle to the opposite side meeting at 90˚.
AD is an altitude.

(1) Find m_{BC}
(2) Find m_{AD} (perpendicular to BC)
(3) Use A (point on line) and m_{AD} to find equation.



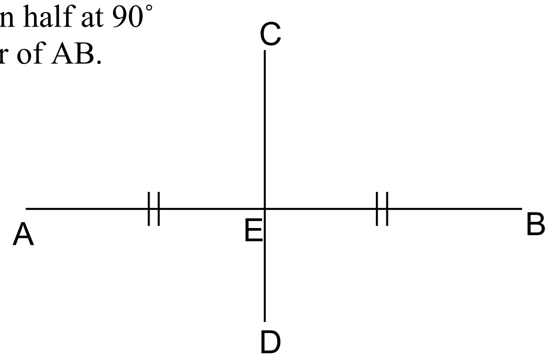
Median From one corner of a triangle to the middle of the opposite side.
AD is a median.

(1) Find D, the midpoint BC.
(2) Find m_{AD}
(3) Use A or D (point on line) and m_{AD} to find equation.



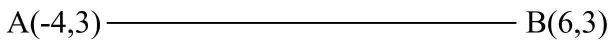
Perpendicular Bisector A line which cuts another line in half at 90˚
CD is the perpendicular bisector of AB.

(1) Find E, the midpoint of AB
(2) Find m_{AB}
(3) Find m_{CD} (perpendicular to AB)
(4) Use E (point on line) and m_{CD} to find equation

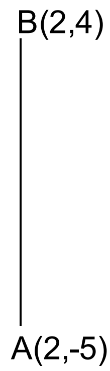


Important: (i) To find a point of contact or intersection of two lines, **solve simultaneously.**

(ii) AB has equation y = 3



(iii) AB has equation x = 2



Functions

(i) If f(g(x)) = x then f(x) and g(x) are **inverses** of each other.

(ii) If f(x) and g(x) are inverse of each other then their graphs are images reflected in the line y = x.

(iii) The **domain** of a function is the set of values the function can take. Usually when asked to state the domain, you will state values x **cannot** be.

e.g. $f(x) = \dfrac{3}{(x-1)(x+3)}$ domain is $x \neq 1, -3 \quad x \in R$

x cannot be 1 or –3 since this gives 0 on the bottom line.

Graphs

y = f(x) + 3	move graph up by 3 (down if –3)
y = f(x + 3)	move graph left by 3 (right if –3)
y = -f(x)	reflect graph in x-axis
y = f(-x)	reflect graph in y-axis
y = 3f(x)	stretch graph vertically by multiple of 3 (compress if fraction)
y = f(3x)	compress horizontally by multiple of 3 (stretch if fraction)

Trigonometry

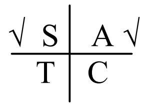
Π radians = 180° i.e. $\frac{\pi}{2} = 90^\circ \quad \frac{\pi}{4} = 45^\circ \quad \frac{2\pi}{3} = 120^\circ$

Exact values

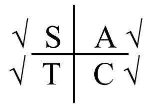
	30^0	45^0	60^0
Sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
Cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
Tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Equations

(i) $\sin 2x = 0.5 \quad 0 \leq x \leq 360$ 2 rotations of



(ii) $\tan^2 x = 9 \quad 0 \leq x \leq 360 \quad \tan x = -3, 3$

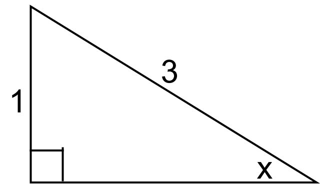


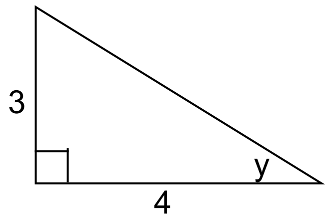
(iii) $\tan^2 x - 2\tan x - 8 = 0 \quad 0 \leq x \leq 360 \quad \rightarrow (\tan x - 4)(\tan x + 2) = 0$
Do as separate trig equations

Addition Formulae

(i) $\sin(A + B) = \sin A\cos B + \cos A\sin B$
(ii) $\sin(A - B) = \sin A\cos B - \cos A\sin B$
(iii) $\cos(A + B) = \cos A\cos B - \sin A\sin B$
(iv) $\cos(A - B) = \cos A\cos B + \sin A\sin B$

<u>Example</u> Given $\sin x = \frac{1}{3}$ and $\tan y = \frac{3}{4}$. Find $\cos(x - y)$



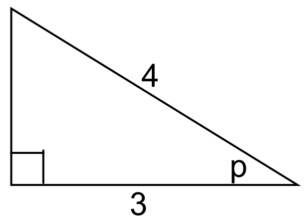


(1) **Draw 2 right-angled triangles**
(2) **Find missing side in each triangle (use Pythagoras)**
(3) **Expand cos(x - y)**
(4) **Substitute values using triangles.**

Double angle formulae

(i) $\sin 2A = 2\sin A\cos A$
(ii) $\cos 2A = 2\cos^2 A - 1$ **or**
$1 - 2\sin^2 A$ **or**
$\cos^2 A - \sin^2 A$

<u>Example 1</u> Given $\cos P = \frac{3}{4}$ find $\cos 2P$



(1) **Draw a right angled triangle**
(2) **Find missing side in triangle (use Pythagoras)**
(3) **Substitute values using triangle**

<u>Example 2</u> Solve $\cos 2x - 4\sin x + 5 = 0$

Replace **cos 2x** by **$1 - 2\sin^2 x$** since equation contains sin x term.
Then factorise and solve as two separate trig equations.

Recurrence relations

$\mathbf{u_{n+1} = 0.6u_n + 10}$ If left hand side is $\mathbf{u_2}$ then right hand side **must** be $\mathbf{u_1}$
If left hand side is $\mathbf{u_4}$ then right hand side **must** be $\mathbf{u_3}$

$u_{n+1} = \mathbf{a}u_n + \mathbf{b}$ You will be given three values for u_n e.g. $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$

$u_3 = \mathbf{a}u_2 + \mathbf{b}$
$u_2 = \mathbf{a}u_1 + \mathbf{b}$

Replace these values as shown opposite, then solve simultaneously to find **a** and **b**.

Limit of a relation

$\mathbf{u_{n+1} = au_n + b}$ has a limit if $\mathbf{-1 < a < 1}$. To find limit use $\mathbf{L} = \frac{b}{1-a}$

Differentiation

Differentiation (finding a derivative) is used to obtain

(i) rate of change
(ii) speed
(iii) gradient of a curve

Equation of a tangent (For tangents to a circle see section on circles)

A tangent is a straight line which touches a curve at a single point.
To find the equation of a tangent

(1) **Given an x value, substitute in function to find y. You now have a point.**
(2) **Differentiate and substitute x in derivative. You now have the gradient.**
(3) **Find equation of tangent using point and gradient.**

Increasing / decreasing functions

(i) If a function is **never decreasing** its gradient **cannot be negative**.
i.e. differentiate and show the derivative involves a squared term
e.g. $\frac{dy}{dx} = (3x + 1)^2$ which cannot be negative

(ii) If a function is **never increasing** its gradient **cannot be positive**.
i.e. differentiate and show the derivative involves a negative squared term e.g. $\frac{dy}{dx} = -(3x + 1)^2$ which cannot be positive

Stationary points

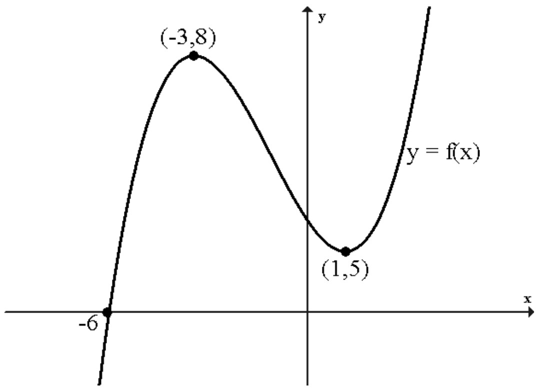
These are points on a curve where the gradient (i.e. the derivative) is 0.
To find a stationary point, **differentiate** and put equal to 0.
Stationary points come as 3 types:

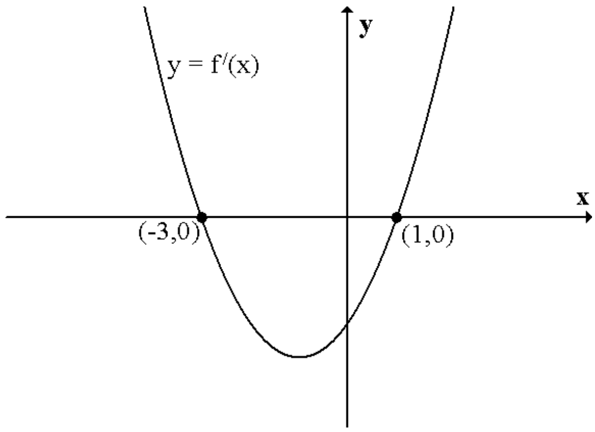
(1) Maximum turning point
(2) Minimum turning point
(3) Point of inflexion

Decide on the nature of a stationary point by drawing a nature table.

Drawing $f'(x)$

Move turning (stationary) points onto the x-axis and consider gradient before and after each turning point. e.g. a turning point of (3,5) moves to (3,0).
Note: Ignore points on the x-axis of the original curve.





Optimisation

If a question gives you a function (or equation) and asks for the biggest, smallest, greatest, least, maximum, minimum, etc. do as a stationary points question i.e. **differentiate and put equal to 0.**

Polynomials

To factorise any polynomial involving a power of x^3 or higher, use **Synthetic division.**
e.g. **given** $\mathbf{x^3 - 6x^2 + 5x + 12}$

If $(x - 3)$ is a factor then dividing by 3 will give a remainder of 0.

3	1	-6	5	12
		3	-9	-12
	1	-3	-4	**0**

i.e. $x^3 - 6x^2 + 5x + 12 = (x - 3)(x^2 - 3x - 4) \rightarrow (x - 3)(x - 4)(x + 1)$

e.g. **given** $\mathbf{x^3 - 6x^2 + 5x + 12 = 0}$

If 3 is a root (or zero), then dividing by 3 will give a remainder of 0. **Divide as above**.

i.e. $x^3 - 6x^2 + 5x + 12 = 0 \rightarrow (x - 3)(x^2 - 3x - 4) = 0 \rightarrow (x - 3)(x - 4)(x + 1) = 0$
$x = 3, 4, -1$

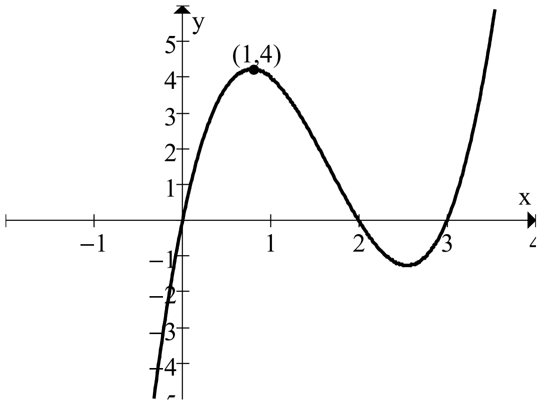
Note: If the factor is of a form similar to $(2x - 1)$ then you must divide by ½ since the the bracket is equivalent to $(x - ½)$.

Functions from graphs

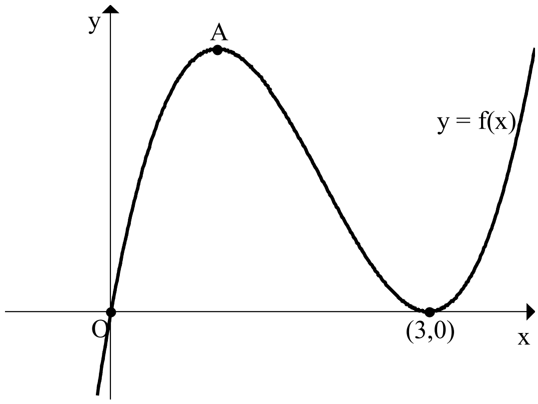
To find a formula for f(x) we set up the equation

$y = kx(x - 2)(x - 3)$

k is included since the curve may have been stretched vertically. Another point on the curve **must** be given to enable you to find k. Find k by substituting this point. Once this has been done expand the brackets.



Important: If f(x) has a turning point on the x-axis this **must** be represented by 2 identical brackets.
i.e. **$y = kx(x - 3)(x - 3)$**



Quadratic Functions

Completing the square

You will be asked to express in the form **$(x + b)^2 + c$** or something similar.

$x^2 + 4x + 7 \rightarrow (x^2 + 4x + 4 - 4) + 7 \rightarrow (x + 2)^2 + 3$

$3x^2 + 12x + 5 \rightarrow 3(x^2 + 4x) + 5 \rightarrow 3(x^2 + 4x + 4 - 4) + 5 \rightarrow 3(x + 2)^2 - 7$

Important: for an expression like $3x^2 + 12x + 5$ you **must** take out a common factor so that you work with only x^2 **not** $3x^2$. Same rule for a term of $-x^2$

Discriminant: Can be used in any context where an x^2 term is involved as the highest power.

$b^2 - 4ac > 0$ roots are real and unequal
$b^2 - 4ac = 0$ roots are real and equal
$b^2 - 4ac < 0$ roots are unreal

(1) To show roots are equal, show that $b^2 - 4ac = 0$
(2) To show roots are always real, show that $b^2 - 4ac \geq 0$
i.e. answer should involve a squared term e.g. $(x + 4)^2$ which cannot be negative.

Tangents

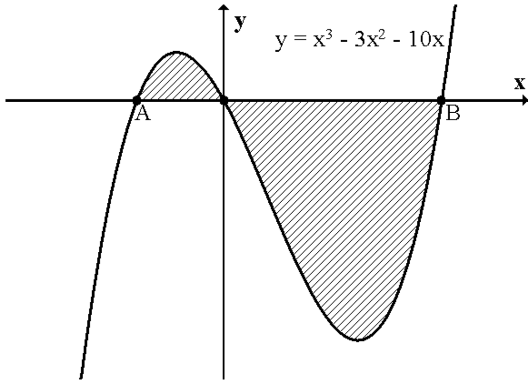
To prove a line is a tangent to a curve (i.e. touch at a single point) put the equations equal to each other and then show $b^2 - 4ac = 0$ (only one solution)

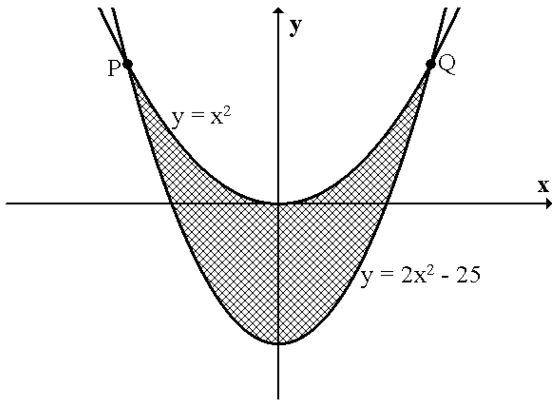
Integration

The opposite process to differentiation. Integration finds the **area between a curve and the x-axis** or **between two curves.**

Note Integrals above the x-axis are positive. Integrals below the x-axis are negative but are changed to positive since a negative area does not exist.

These areas must be done separately since the area below the x-axis will be negative.





Integral between 2 curves should be

top curve – bottom curve.

i.e. $\mathbf{x^2 - (2x^2 - 25)}$

Simplify before integrating.

Important Limits are the **x-values** at either end of the area.
If these are not given solve simultaneously to find them.

Note When calculating an indefinite integral you **must** include a constant C.
e.g. $\int 3x^2\,dx = \mathbf{x^3 + C}$

Differential Equations

Given an equation in terms of $f^{/}(x)$ or $\frac{dy}{dx}$, integrate and remember to include + C.
You will also be given an extra point which you can substitute to enable you to find C.
Finish by rewriting equation.

Example $f^{/}(x) = 3x^2 - 4x + 1$ and $f(2) = 5$. Find a formula for f(x).

$f(x) = x^3 - 2x^2 + x + C$
$5 = 2^3 - 2 \times 2^2 + 2 + C$
$3 = C$ i.e. formula is $f(x) = x^3 - 2x^2 + x + 3$

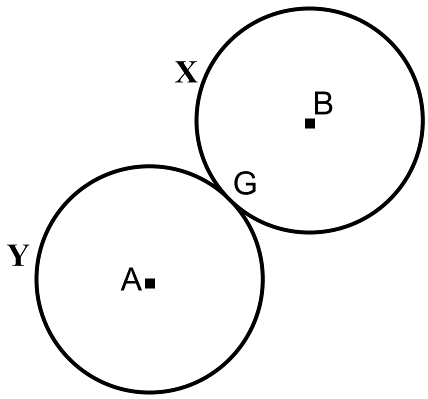
Circles

(i) $x^2 + y^2 = r^2$ is a circle centre the origin and radius r
(ii) $(x - a)^2 + (y - b)^2 = r^2$ is a circle centre (a,b) and radius r.
(iii) $x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle centre (-g,-f) and radius $\sqrt{g^2 + f^2 - c}$

Notes

(1) Equations (ii) and (iii) are really the same equation, (ii) has the brackets expanded.
(2) The equation **must** be $x^2 + y^2$ and **not** $x^2 - y^2$ or $x^2 + 3y^2$ etc.
(3) To make up the equation of a circle given radius and centre you **must** use equation (ii).
(4) If $g^2 + f^2 - c < 0$ then the circle does not exist since the radius would be negative.

Touching Circles



To show that 2 circles touch:
radius X + radius Y = distance between the centres.

Use Pythagoras to find distance from A to B.

Note
If radius X + radius Y $>$ distance AB then circles intersect
If radius X + radius Y $<$ distance AB then circles do not touch.

Intersection of line and circle

Substitute equation of line into equation of circle and solve.

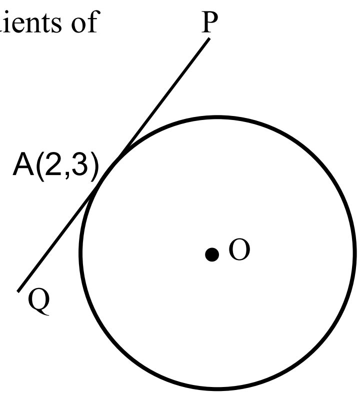
e.g. line $y = x + 3$ and circle $x^2 + y^2 + 4x - 8y + 11 = 0$
becomes $x^2 + (x + 3)^2 + 4x - 8(x + 3) + 11 = 0$ etc.

Proving tangency

As above i.e. substitute equation of line into equation of circle.
If line is a tangent then there is only one point of contact. We can use $\mathbf{b^2 - 4ac = 0}$ to prove only one solution or solve and show only one answer.
It is better, however, to factorise and show that you only get one solution. **Make sure you write a statement saying it is a tangent since there is only one solution.**

Equation of a tangent to a circle

Since a tangent will meet a radius of the circle at 90° we can use gradients of perpendicular lines.
To find equation of tangent



(1) Find centre of circle

(2) Find gradient of radius OA using $\mathbf{m} = \frac{y_2 - y_1}{x_2 - x_1}$

(3) Write down gradient of tangent PQ

(4) Use point A and $\mathbf{m_{PQ}}$ to find equation

Wave Function:

You will be asked to express something in the form kcos(x + a), kcos(x – a), ksin(x – a) or ksin(x + a).

(1) Use the addition formula to expand which of the above you are given
(2) Match the question with this expansion
(3) Find k. Do as if you are doing a Pythagoras question.

(4) Find a. Use the fact tan a = $\frac{\sin a}{\cos a}$. You must check which quadrant a is in.

Example Express 2cos x + sin x in the form kcos (x – a) where k > 0 and $0 \le a \le 360$

(1) kcos (x – a) = kcos x cos a + ksin x sin a
(2) = 2cos x + sin x i.e. kcos a = 2 ksin a = 1

(3) $k^2 = 2^2 + 1^2$ (always add)
$k = \sqrt{5}$

(4) tan a = ½
$a = 26.6^0$

s	a √√
t	c

cos a and sin a both positive i.e. answer in first quadrant.

Therefore 2cos x + sin x = $\sqrt{5}\cos(x - 26.6)^0$.

Note: If expression had been 2cos 2x + sin 2x the same process would be followed but your answer would be in the form $\sqrt{5}\cos(2x - 26.6)^0$.

You may then be asked to do one of three things:

(1) Solve an equation.
For example solve 2cos x + sin x = 1. Solve as a simple trig equation.

(2) **Find a maximum or minimum value and the corresponding value of x.**
For the example above:

$\sqrt{5}\cos(x - 26.6)^0$ has maximum of $\sqrt{5}$ when x – 26.6 = 0 i.e. x = 26.6

$\sqrt{5}\cos(x - 26.6)^0$ has minimum of $-\sqrt{5}$ when x – 26.6 = 180 i.e. x = 206.6

The value of x is decided by thinking of where the cos graph has a maximum (0^0) or minimum (180^0) value.

(3) Draw a graph.
For the example above

$\sqrt{5}\cos(x - 26.6)^0$ would be stretched to $\sqrt{5}$ and $-\sqrt{5}$ and moved 26.6^0 to the right.

Make sure you also find the point where the new graph cuts the y-axis (x = 0).

Vectors

All the following rules and processes work in both 2 and 3 dimensions.

Position vectors:

The position vector of a point is the vector formed when the origin is joined to that point.

For example, if A has coordinates (2,1,-4), its position vector has components $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$

Unit vectors:

A unit vector is a vector one unit long.

There are three important unit vectors $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

This means a vector $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ would have components $\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$

Important: Given points A and B the vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
Given points C and D the vector $\overrightarrow{CD} = \mathbf{d} - \mathbf{c}$

Magnitude:

The magnitude of a vector is its length. The magnitude of a vector **a** is written as $|\mathbf{a}|$ and is found by using Pythagoras.

Example: If $\mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ then $|\mathbf{a}| = \sqrt{2^2 + -2^2 + 1^2}$ $\quad |\mathbf{a}| = 3$

Note: If you are asked to find a unit vector parallel to a given vector, find the magnitude of the given vector and then divide each component by this magnitude.

A unit vector parallel to the vector **a** above would have components $\begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$

Scalar Product:

The scalar product of two vectors **a** and **b** is defined as

$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ given components **or**
$|\mathbf{a}|\,|\mathbf{b}|\cos\theta$ given magnitudes and an angle

(1) If the scalar product is used the vectors **must** travel away from a common point.

(2) **Important result:** If $\mathbf{a}.\mathbf{b} = 0$ then the vectors **a** and **b** are **perpendicular.**

(3) Given something of the form **a.(a + b)** expand as **a.a** + **a.b** and remember

$$\mathbf{a.a} = |\mathbf{a}|^2 \qquad \mathbf{b.b} = |\mathbf{b}|^2 \quad \text{etc.}$$

Collinearity:

If 3 points A, B and C are collinear they lie on the same straight line. (See equation of a line).

To prove A, B and C are collinear

(1) **Find the vectors** $\overrightarrow{AB}$ **and** $\overrightarrow{BC}$
(2) **Show they are multiples of the same vector**
(3) **Write a statement: Since** $\overrightarrow{AB}$ **and** $\overrightarrow{BC}$ **are parallel and B is a common point A, B and C are collinear.**

Section Formula:

This is used to find the coordinates of a point which divides two other points in a given ratio.

Example: A is the point (2,-1,6) and C is (5,-4,0). The point B divides AC in the ratio 2:1. Find the coordinates of B.

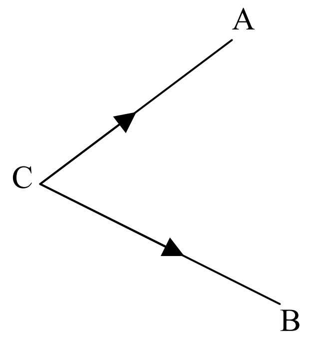
(1) **Write A and C in component form**
(2) **Write given ratio underneath**
(3) **Cross-multiply and divide by sum of ratios**
(4) **Write final answer as coordinates**

$$\underset{2}{\overset{A}{\begin{pmatrix}2\\-1\\6\end{pmatrix}}} \quad \underset{1}{\overset{C}{\begin{pmatrix}5\\-4\\0\end{pmatrix}}} \quad = \frac{\begin{pmatrix}2\\-1\\6\end{pmatrix}+\begin{pmatrix}10\\-8\\0\end{pmatrix}}{3} = \begin{pmatrix}4\\-3\\2\end{pmatrix} \quad B(4,-3,2)$$

Angle between two vectors:

To find the angle between two vectors use $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}||\mathbf{b}|}$

Given coordinates of A, B and C, find the size of angle ACB.



(1) **Draw required angle. Make sure vectors travel away from angle.**
(2) **Calculate required vectors i.e.** $\overrightarrow{CA}$ **and** $\overrightarrow{CB}$
(3) **Calculate** $|\overrightarrow{CA}|$ **and** $|\overrightarrow{CB}|$
(4) **Substitute answers into formula above and evaluate.**

Note: If calculation results in a negative answer, angle is between 90^0 and 180^0.

Further Calculus

All previous rules for integration and differentiation still apply. For example, in terms of

Stationary points
Equations of tangents
Increasing / Decreasing functions
Area under a curve or between two curves
Differential equations

The Chain Rule:

Differentiation

$f(x) = (ax + b)^n \qquad f'(x) = n(ax + b)^{n-1} \times a$

i.e. Differentiate as x^n then multiply by the derivative of the term in the bracket

e.g. $f(x) = (6x - 2)^4 \qquad f'(x) = 4(6x - 2)^3 \times 6$

Integration

$\int (ax + b)^n = \dfrac{(ax + b)^{n+1}}{a(n+1)} + C$

i.e. Integrate as x^n then divide also by derivative of the term in the bracket

$\int (6x - 2)^4 \, dx = \dfrac{(6x - 2)^5}{5 \times 6} + C$

Differentiation

$f(x) = \sin x \qquad f'(x) = \cos x$

$f(x) = \cos x \qquad f'(x) = -\sin x$

Integration

$\int \sin x \, dx = -\cos x + C$

$\int \cos x \, dx = \sin x + C$

These rules are given in the examination.

For more complicated functions use the chain rule.

Differentiation

$f(x) = 4\sin 3x \qquad f'(x) = 4\cos 3x \times 3$

$= 12\cos 3x$

$f(x) = 4\sin^2 x \qquad f'(x) = 8\sin x \cos x$

For the above example, differentiate as $4x^2$ then multiply by the derivative of sin x

Integration

$\int 4\sin 3x \, dx = \dfrac{-4\cos 3x}{3} + C$

Integrals of the form $\int 4\sin^2 x \, dx$ cannot be asked at Higher level

Very important:

If differentiating or integrating trig functions i.e. those involving sin or cos, you <u>must</u> work in radians. Having said this, if you are given angles in the form $\frac{\pi}{6}, \frac{2\pi}{3}$ you can change to degrees and do your calculations in degrees.

Any other values, e.g. 1,3, etc., will be in radians.
The simplest way of approaching this is to change your calculator setting to radians and just do the calculations as normal.

Examples:

(1) Find the equation of the tangent to the curve $y = 4\cos\left(2x - \frac{\pi}{6}\right)$ at the point where $x = \frac{\pi}{2}$

Solution: at $x = \frac{\pi}{2}$ $y = 4\cos(2 \times 90^0 - 30^0) = 4 \times -\frac{\sqrt{3}}{2}$ i.e. $y = -2\sqrt{3}$

$$\frac{dy}{dx} = -8\sin\left(2x - \frac{\pi}{6}\right) \quad \text{at } x = \frac{\pi}{2} \quad \frac{dy}{dx} = -8\sin(2 \times 90^0 - 30^0) = -8 \times \frac{1}{2} = -4$$

Equation is $y + 2\sqrt{3} = -4(x - \frac{\pi}{2})$

(2) $$\int_1^3 \cos 2x \, dx = \left[\frac{\sin 2x}{2}\right]_1^3 = \left(\frac{\sin 6}{2}\right) - \left(\frac{\sin 2}{2}\right) = -0.14 - 0.45 = -0.59$$

Remember if this is an area the answer would be changed to a positive value.

Logarithms

Logarithms are simply powers of a given number. This number is called the base.
For example

$\log_{10} 1000 = 3$ The base in this case is 10 and the value is 3 since 1000 is 10^3

$\log_2 16 = 4$ The base in this case is 2 and the value is 4 since $2^4 = 16$

Laws of logarithms:

(1) If $\log_a x = b$ then $x = a^b$. **This is the most important law.**

For example $\log_{10} 100 = 2$ gives $100 = 10^2$

(2) $\log_a x + \log_a y = \log_a xy$

For example $\log_{10} 4 + \log_{10} 25 = \log_{10} 100 = 2$

(3) $\log_a x - \log_a y = \log_a \frac{x}{y}$

For example $\log_3 54 - \log_3 2 = \log_3 27 = 3$ (since $27 = 3^3$)

(4) $\log_a x^n = n\log_a x$

Notes:
(1) $\mathbf{\log_a 1 = 0}$, no matter the base. This is because anything to the power 0 is 1 i.e. $a^0 = 1$.

(2) Calculations of the types above require the logarithms to have the same base.
The most common bases are (i) base 10, this is the log button on your calculator
(ii) base e, this is the ln button on your calculator
base e is called natural logarithms

Logarithmic Equations

Example: $\log_2 x + \log_2 (x - 2) = 3$

(1) **Make sure each part of the equation is a logarithm**

3 must be expressed as a log
i.e. $3 = \log_2 8$
equation is $\log_2 x + \log_2(x - 2) = \log_2 8$

(2) **Use laws of logs to simplify** $\log_2 x(x - 2) = \log_2 8$

(3) **Drop the logs** $x(x - 2) = 8$

(4) **Solve the resultant equation** $x^2 - 2x - 8 = 0$

$(x - 4)(x + 2) = 0$

(5) **Answer must be positive** $x = 4$

Exponential growth and decay

Questions will almost certainly involve logarithms to the base e.

For exponential growth the formula will be of the form $\mathbf{N = N_o e^{kt}}$

For exponential decay the formula will be of the form $\mathbf{N = N_o e^{-kt}}$

Where $\mathbf{N_o}$ is the original value and **N** is the value after a period of time **t**.

Strategy

(1) **Substitute given values. Use normal rules of equations to simplify as much as possible.**
(2) **After simplification, if power part of the equation is a number, do not take logs, just work out on the calculator.**
(3) **After simplification, if power part of the equation involves a letter, take logs of both sides. If equation involves e, take log base e (ln on calculator). If equation involves base 10 take log base 10 (log on calculator).**
Use the fact $\log_e e = 1$ or $\log_{10} 10 = 1$.
(4) **For questions involving half-life: Given $N = N_o e^{-kt}$ replace N by $\frac{1}{2}N_o$ to give**

$$\tfrac{1}{2}N_o = N_o e^{-kt}$$

$$\tfrac{1}{2} = e^{-kt}$$ **then take logs of both side**

Example:

For a radioactive substance $A = A_o e^{-kt}$, where A_o is the original mass and t is the time in minutes. In 5 minutes, 20g of this substance is reduced to 16g.

(a) Find k to 2 significant figures.
(b) Find the half life of this substance.

Solution:

(a)

(1) **Substitute given values** $16 = 20e^{-k \times 5}$

(2) **Simplify** $0.8 = e^{-5k}$ $(16 \div 20 = 0.8)$

(3) **Letter as power i.e. take logs** $\log_e 0.8 = \log_e e^{-5k}$

(4) **Use laws of logs** $-0.223 = -5k \log_e e$ remember $\log_e e = 1$

$0.045 = k$ i.e formula becomes $A = A_o e^{-0.045t}$

(b)

(1) **Replace A by $\frac{1}{2}A_o$** $\frac{1}{2}A_o = A_o e^{-0.045t}$

(2) **Simplify and take logs of both sides** $\log_e \frac{1}{2} = \log_e e^{-0.045t}$

(3) **Use laws of logs** $\log_e \frac{1}{2} = -0.045t \log_e e$

(4) **Evaluate $\log_e \frac{1}{2}$** $-0.693 = -0.045t$ $t = 15.4$ minutes

Experimental data

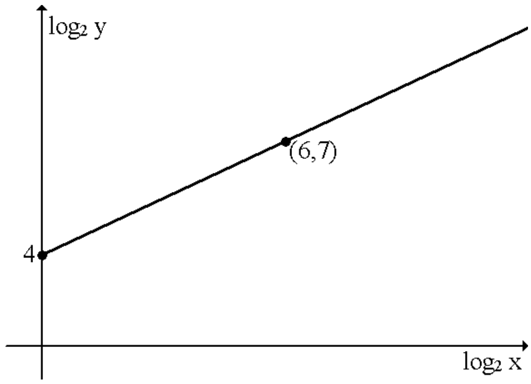
This can be in two forms. You will be told which to use in the examination.

$y = kx^n$

If a relation is of this form it can be plotted on a graph of $\log_a y$ against $\log_a x$ to give a straight line.

Example:

The graph opposite illustrates the law $y = kx^n$.
Find the values of k and n.



Solution:

(1) **Find equation of line in terms of x and y** $m = \frac{7-4}{6-0} = \frac{1}{2}$ i.e. $y = \frac{1}{2}x + 4$

(2) **Rewrite using $\log_2 y$ and $\log_2 x$** $\log_2 y = \frac{1}{2}\log_2 x + 4$

(3) **Convert 4 into a log** $\log_2 y = \frac{1}{2}\log_2 x + \log_2 16$

(4) **Use laws of logs to simplify** $\log_2 y = \log_2 x^{½} + \log_2 16$

$\log_2 y = \log_2 16x^{½}$

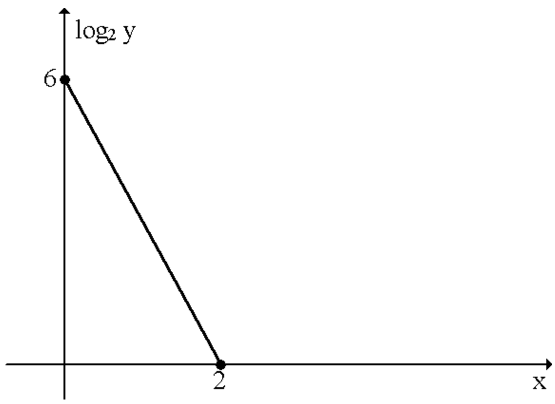
(5) **Drop the logs** $y = 16x^{½}$ i.e. $k = 16$ $n = ½$

$y = ab^x$

If a relation is of this form it can be plotted on a graph of $\log_a y$ against x to give a straight line.

Example:

The graph opposite illustrates the law $y = ab^x$.
Find the values of a and b.



Solution:

(1) **Find equation in terms of x and y** $y = -3x + 6$

(2) **Rewrite using $\log_2 y$ and x** $\log_2 y = -3x + 6$

(3) **Convert 3 and 6 into logs** $\log_2 y = -x\log_2 8 + \log_2 64$

(4) **Use laws of logs to simplify** $\log_2 y = \log_2 8^{-x} + \log_2 64$

$\log_2 y = \log_2 64(8^{-x})$

(5) **Drop the logs** $y = 64(8^{-x})$ i.e. $a = 64$ $b = -8$

Logarithmic Graphs:

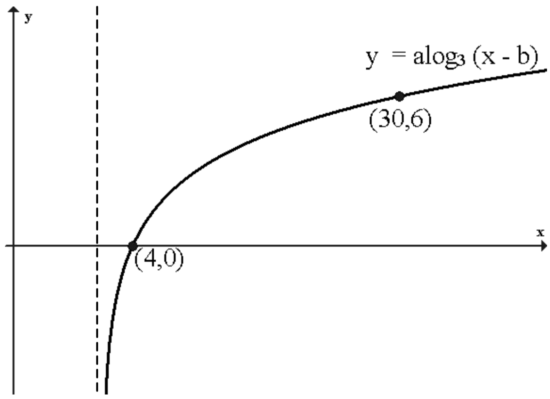
A logarithmic graph is the inverse of an exponential graph.
You will be asked to do one of two things with a logarithmic graph.

(1) Find unknowns given points on the graph.
(2) Use rules of graphs to move the graph in some way. **See Functions.**

Finding Unknowns

Example:

The diagram shows the graph of $y = a\log_3(x - b)$.
Find a and b.



Solution:

(1) **Always substitute point on x-axis first**

$y = a\log_3(x - b)$
$0 = a\log_3(4 - b)$
$0 = \log_3(4 - b)$ dividing by a

(3) **Use laws of logs to simplify and solve**

$3^0 = 4 - b$ i.e. $b = 3$

(3) **Replace this answer in the formula**

$y = a\log_3(x - 3)$

(4) **Substitute second point**

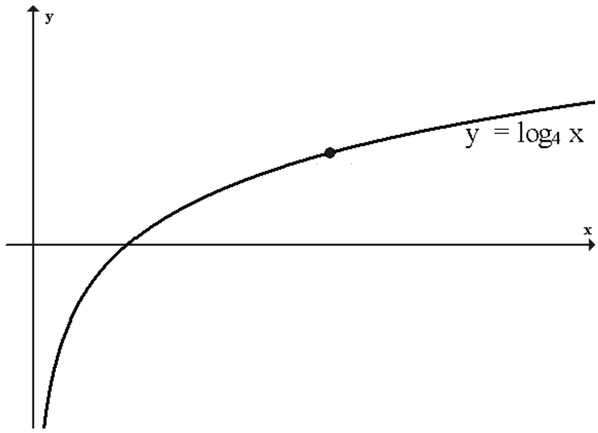
$6 = a\log_3(30 - 3)$
$6 = a\log_3 27$
$6 = a \times 3$ i.e. $a = 2$

Moving Graphs

Moving logarithmic graphs follows the same rules as for any other graphs.
However, it is sometimes necessary to express the graph in the correct form.

Examples

The diagram shows part of the graph of $y = \log_4 x$.



(a) Sketch $y = \log_4(x - 3)$
Move graph 3 to the right

(b) Sketch $y = \log_4 16x$
Using laws of logs $y = \log_4 16 + \log_4 x$
$y = 2 + \log_4 x$
Move graph up by 2

(c) Sketch $y = \log_4 \frac{1}{x^2}$
Using laws of logs $y = \log_4 x^{-2}$
$y = -2\log_4 x$
Reflect graph in x-axis and then stretch vertically by 2